OCR

RECOGNISING ACHIEVEMENT

ADVANCED GCE **4727/01**

MATHEMATICS

Further Pure Mathematics 3

THURSDAY 24 JANUARY 2008 Morning

Time: 1 hour 30 minutes

Additional materials: Answer Booklet (8 pages)
List of Formulae (MF1)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- **You are reminded of the need for clear presentation in your answers.**

This document consists of **4** printed pages.

1 **(a)** A group G of order 6 has the combination table shown below.

	e	a	b	p	q	r
e	e	a	b	p	q	r
a	a	b	e	r	p	q
b	b	e	a	q	r	p
p	p	q	r	e	a	b
q	q	r	p	b	e	a
r	r	p	q	a	b	e

(i) State, with a reason, whether or not G is commutative. [1]

(ii) State the number of subgroups of G which are of order 2. [1]

(iii) List the elements of the subgroup of G which is of order 3. [1]

(b) A multiplicative group H of order 6 has elements e, c, c^2, c^3, c^4, c^5, where e is the identity. Write down the order of each of the elements c^3, c^4 and c^5. [3]

2 Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 4x. \quad [7]$$

3 Two fixed points, A and B, have position vectors $\mathbf{a}$ and $\mathbf{b}$ relative to the origin O, and a variable point P has position vector $\mathbf{r}$.

(i) Give a geometrical description of the locus of P when $\mathbf{r}$ satisfies the equation $\mathbf{r} = \lambda\mathbf{a}$, where $0 \leqslant \lambda \leqslant 1$. [2]

(ii) Given that P is a point on the line AB, use a property of the vector product to explain why $(\mathbf{r} - \mathbf{a}) \times (\mathbf{r} - \mathbf{b}) = \mathbf{0}$. [2]

(iii) Give a geometrical description of the locus of P when $\mathbf{r}$ satisfies the equation $\mathbf{r} \times (\mathbf{a} - \mathbf{b}) = \mathbf{0}$. [3]

4 The integrals C and S are defined by

$$C = \int_0^{\frac{1}{2}\pi} e^{2x} \cos 3x \, dx \qquad \text{and} \qquad S = \int_0^{\frac{1}{2}\pi} e^{2x} \sin 3x \, dx.$$

By considering $C + iS$ as a single integral, show that

$$C = -\tfrac{1}{13}(2 + 3e^{\pi}),$$

and obtain a similar expression for S. [8]

(You may assume that the standard result for $\int e^{kx}\,dx$ remains true when k is a complex constant, so that $\int e^{(a+ib)x}\,dx = \dfrac{1}{a+ib}e^{(a+ib)x}$.)

5 **(i)** Find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = \sin 2x,$$

expressing y in terms of x in your answer. [6]

In a particular case, it is given that $y = \dfrac{2}{\pi}$ when $x = \frac{1}{4}\pi$.

(ii) Find the solution of the differential equation in this case. [2]

(iii) Write down a function to which y approximates when x is large and positive. [1]

6 A tetrahedron $ABCD$ is such that AB is perpendicular to the base BCD. The coordinates of the points A, C and D are $(-1, -7, 2)$, $(5, 0, 3)$ and $(-1, 3, 3)$ respectively, and the equation of the plane BCD is $x + 2y - 2z = -1$.

(i) Find, in either order, the coordinates of B and the length of AB. [5]

(ii) Find the acute angle between the planes ACD and BCD. [6]

7 **(i)** **(a)** Verify, without using a calculator, that $\theta = \frac{1}{8}\pi$ is a solution of the equation $\sin 6\theta = \sin 2\theta$. [1]

(b) By sketching the graphs of $y = \sin 6\theta$ and $y = \sin 2\theta$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, or otherwise, find the other solution of the equation $\sin 6\theta = \sin 2\theta$ in the interval $0 < \theta < \frac{1}{2}\pi$. [2]

(ii) Use de Moivre's theorem to prove that

$$\sin 6\theta \equiv \sin 2\theta(16\cos^4\theta - 16\cos^2\theta + 3). \qquad [5]$$

(iii) Hence show that one of the solutions obtained in part **(i)** satisfies $\cos^2\theta = \frac{1}{4}(2 - \sqrt{2})$, and justify which solution it is. [3]

8 Groups A, B, C and D are defined as follows:

A: the set of numbers $\{2, 4, 6, 8\}$ under multiplication modulo 10,

B: the set of numbers $\{1, 5, 7, 11\}$ under multiplication modulo 12,

C: the set of numbers $\{2^0, 2^1, 2^2, 2^3\}$ under multiplication modulo 15,

D: the set of numbers $\left\{\frac{1+2m}{1+2n}, \text{ where } m \text{ and } n \text{ are integers}\right\}$ under multiplication.

(i) Write down the identity element for each of groups A, B, C and D. [2]

(ii) Determine in each case whether the groups

A and B,

B and C,

A and C

are isomorphic or non-isomorphic. Give sufficient reasons for your answers. [5]

(iii) Prove the closure property for group D. [4]

(iv) Elements of the set $\left\{\frac{1+2m}{1+2n}, \text{ where } m \text{ and } n \text{ are integers}\right\}$ are combined under **addition**. State which of the four basic group properties are **not** satisfied. (Justification is not required.) [2]

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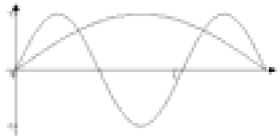
1 (a) (i) e.g. $ap \neq pa \Rightarrow$ not commutative	B1 **1**	For correct reason and conclusion
(ii) 3	B1 **1**	For correct number
(iii) e, a, b	B1 **1**	For correct elements
(b) c^3 has order 2 c^4 has order 3 c^5 has order 6	B1 B1 B1 **3** **6**	For correct order For correct order For correct order
2 $m^2 - 8m + 16 = 0$ $\Rightarrow m = 4$ $\Rightarrow$ CF $(y =)\ (A + Bx)e^{4x}$ For PI try $y = px + q$ $\Rightarrow -8p + 16(px + q) = 4x$ $\Rightarrow p = \frac{1}{4} \quad q = \frac{1}{8}$ $\Rightarrow$ GS $y = (A + Bx)e^{4x} + \frac{1}{4}x + \frac{1}{8}$	M1 A1 A1√ M1 A1 A1 B1√ **7** **7**	For stating and attempting to solve auxiliary eqn For correct solution For CF of correct form. f.t. from m For using linear expression for PI For correct coefficients For GS = CF + PI. Requires $\boxed{y =}$. f.t. from CF and PI with 2 arbitrary constants in CF and none in PI
3 (i) line segment OA	B1 B1 **2**	For stating line through O *OR* A For correct description **AEF**
(ii) $(\mathbf{r} - \mathbf{a}) \times (\mathbf{r} - \mathbf{b}) = \overrightarrow{AP} \times \overrightarrow{BP}$ $= \lvert AP \rvert \lvert BP \rvert \sin \pi \,.\, \hat{\mathbf{n}} = \mathbf{0}$	B1 B1 **2**	For identifying $\mathbf{r} - \mathbf{a}$ with $\overrightarrow{AP}$ and $\mathbf{r} - \mathbf{b}$ with $\overrightarrow{BP}$ Allow direction errors For using × of 2 parallel vectors = **0** *OR* sin π = 0 *or* sin 0 = 0 in an appropriate vector expression
(iii) line through O parallel to AB	B1 B1 B1 **3** **7**	For stating line For stating through O For stating correct direction **SR** For $\overrightarrow{AB}$ or $\overrightarrow{BA}$ allow B1 B0 B1
4 $(C + iS =)\ \int_0^{\frac{1}{2}\pi} e^{2x}(\cos 3x + i \sin 3x)(dx)$ $\cos 3x + i \sin 3x = e^{3ix}$ $\int_0^{\frac{1}{2}\pi} e^{(2+3i)x}(dx) = \frac{1}{2+3i}\left[e^{(2+3i)x}\right]_0^{\frac{1}{2}\pi}$ $= \frac{2-3i}{4+9}\left(e^{(2+3i)\frac{1}{2}\pi} - e^0\right) = \frac{2-3i}{13}\left(-i e^{\pi} - 1\right)$ $= \left\{\frac{1}{13}\left(-2 - 3e^{\pi} + i(3 - 2e^{\pi}\right)\right\}$ $C = -\frac{1}{13}\left(2 + 3e^{\pi}\right)$ $S = \frac{1}{13}\left(3 - 2e^{\pi}\right)$	 B1 M1* A1 A1 M1 (dep*) M1 (dep*) A1 A1 **8**	 For using de Moivre, seen or implied For writing as a single integral in exp form For correct integration (ignore limits) For substituting limits correctly (unsimplified) (may be earned at any stage) For multiplying by complex conjugate of 2+3i For equating real and/or imaginary parts For correct expression **AG** For correct expression

5 (i) IF $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$ *OR* $x\frac{dy}{dx} + y = x\sin 2x$	M1	For correct process for finding integrating factor *OR* for multiplying equation through by x
$\Rightarrow \frac{d}{dx}(xy) = x\sin 2x$	A1	For writing DE in this form (may be implied)
$\Rightarrow xy = \int x\sin 2x(dx)$	M1	For integration by parts the correct way round
$xy = -\frac{1}{2}x\cos 2x + \frac{1}{2}\int\cos 2x(dx)$	A1	For 1st term correct
$xy = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\ (+c)$	M1	For their 1st term and attempt at integration of $\begin{matrix}\cos\\ \sin\end{matrix} kx$
$\Rightarrow y = -\frac{1}{2}\cos 2x + \frac{1}{4x}\sin 2x + \frac{c}{x}$	A1 **6**	For correct expression for y
(ii) $\left(\frac{1}{4}\pi, \frac{2}{\pi}\right) \Rightarrow \frac{2}{\pi} = \frac{1}{\pi} + \frac{4c}{\pi} \Rightarrow c = \frac{1}{4}$	M1	For substituting $\left(\frac{1}{4}\pi, \frac{2}{\pi}\right)$ in solution
$\Rightarrow y = -\frac{1}{2}\cos 2x + \frac{1}{4x}\sin 2x + \frac{1}{4x}$	A1 **2**	For correct solution. Requires $\boxed{y =}$.
(iii) $(y \approx) - \frac{1}{2}\cos 2x$	B1√ **1** $\boxed{9}$	For correct function **AEF** f.t. from **(ii)**
6 (i)		***Either coordinates or vectors may be used*** Methods 1 and 2 may be combined, for a maximum of 5 marks
METHOD 1		
State $B = (-1, -7, 2) + t(1, 2, -2)$	M1	For using vector normal to plane
On plane $\Rightarrow (-1+t) + 2(-7+2t) - 2(2-2t) = -1$	M1 M1	For substituting parametric form into plane For solving a linear equation in t
$\Rightarrow t = 2 \Rightarrow B = (1, -3, -2)$	A1	For correct coordinates
$AB = \sqrt{2^2+4^2+4^2}$ *OR* $2\sqrt{1^2+2^2+2^2} = 6$	A1 **5**	For correct length of AB
METHOD 2		
$AB = \left\lvert\frac{-1-14-4+1}{\sqrt{1^2+2^2+2^2}}\right\rvert = 6$	M1	For using a correct distance formula
OR $AB = \mathbf{AC} \cdot \hat{\mathbf{AB}} = \frac{[6, 7, 1] \cdot [1, 2, -2]}{\sqrt{1^2+2^2+2^2}} = 6$	A1	For correct length of AB
$B = (-1, -7, 2) \pm 6\frac{(1, 2, -2)}{\sqrt{1^2+2^2+2^2}}$	M1	For using $B = A$ + length of AB × unit normal
$B = (-1, -7, 2) \pm (2, 4, -4)$	B1	For checking whether + or – is needed (substitute into plane equation)
$B = (1, -3, -2)$	A1	For correct coordinates (allow even if B0)
(ii) Find vector product of any two of $\pm[6, 7, 1]$, $\pm[6, -3, 0]$, $\pm(0, 10, 1)$	M1	For finding vector product of two relevant vectors
Obtain $k[1, 2, -20]$	A1	For correct vector **n**
$\theta = \cos^{-1}\frac{\lvert[1, 2, -2] \cdot [1, 2, -20]\rvert}{\sqrt{1^2+2^2+2^2}\sqrt{1^2+2^2+20^2}}$	M1* M1 (dep*)	For using scalar product of two normal vectors For stating both moduli in denominator
$\theta = \cos^{-1}\frac{45}{\sqrt{9}\sqrt{405}} = 41.8°\ (41.810...°, 0.72972...)$	A1√ A1 **6** $\boxed{11}$	For correct scalar product. f.t. from **n** For correct angle

7 (i) (a) $\sin\frac{6}{8}\pi = \frac{1}{\sqrt{2}}$, $\sin\frac{2}{8}\pi = \frac{1}{\sqrt{2}}$	B1 **1**	For verifying $\theta = \frac{1}{8}\pi$
(b)  $\theta = \frac{3}{8}\pi$	M1 A1 **2**	For sketching $y = \sin 6\theta$ and $y = \sin 2\theta$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ *OR* any other correct method for solving $\sin 6\theta = \sin 2\theta$ for $\theta \neq k\frac{\pi}{2}$ *OR* appropriate use of symmetry *OR* attempt to verify a reasonable guess for θ For correct θ
(ii) Im $(c + \mathrm{i}s)^6 = 6c^5 s - 20c^3 s^3 + 6cs^5$ $\sin 6\theta = \sin\theta\left(6c^5 - 20c^3(1-c^2) + 6c(1-c^2)^2\right)$ $\sin 6\theta = \sin\theta\left(32c^5 - 32c^3 + 6c\right)$ $\sin 6\theta = 2\sin\theta\cos\theta\left(16c^4 - 16c^2 + 3\right)$ $\sin 6\theta = \sin 2\theta\left(16\cos^4\theta - 16\cos^2\theta + 3\right)$	M1 A1 M1 A1 A1 **5**	For expanding $(c + \mathrm{i}s)^6$; at least 3 terms and 3 binomial coefficients needed For 3 correct terms For using $s^2 = 1 - c^2$ For any correct intermediate stage For obtaining this expression correctly **AG**
(iii) $16c^4 - 16c^2 + 3 = 1$ $\Rightarrow c^2 = \frac{2 \pm \sqrt{2}}{4}$ $-$ sign requires larger $\theta = \frac{3}{8}\pi$	M1 A1 A1 **3** **11**	For stating this equation **AEF** For obtaining both values of c^2 For stating and justifying $\theta = \frac{3}{8}\pi$ Calculator OK if figures seen

8 **(i)** Group A: $e = 6$ Group B: $e = 1$ Group C: $e = 2^0$ *OR* 1 Group D: $e = 1$	B1 B1 **2**	For any two correct identities For two other correct identities **AEF** for D, but not "$m = n$"
(ii) *EITHER* / *OR* $\begin{array}{c\|cccc} A & 2 & 4 & 6 & 8 \\ \hline 2 & 4 & 8 & 2 & 6 \\ 4 & 8 & 6 & 4 & 2 \\ 6 & 2 & 4 & 6 & 8 \\ 8 & 6 & 2 & 8 & 4 \end{array}$ orders of elements 1, 2, 4, 4 *OR* cyclic group $\begin{array}{c\|cccc} B & 1 & 5 & 7 & 11 \\ \hline 1 & 1 & 5 & 7 & 11 \\ 5 & 5 & 1 & 11 & 7 \\ 7 & 7 & 11 & 1 & 5 \\ 11 & 11 & 7 & 5 & 1 \end{array}$ orders of elements 1, 2, 2, 2 *OR* non-cyclic group *OR* Klein group $\begin{array}{c\|cccc} C & 2^0 & 2^1 & 2^2 & 2^3 \\ \hline 2^0 & 2^0 & 2^1 & 2^2 & 2^3 \\ 2^1 & 2^1 & 2^2 & 2^3 & 2^0 \\ 2^2 & 2^2 & 2^3 & 2^0 & 2^1 \\ 2^3 & 2^3 & 2^0 & 2^1 & 2^2 \end{array}$ orders of elements 1, 2, 4, 4 *OR* cyclic group	B1* B1*	For showing group table *OR* sufficient details of orders of elements *OR* stating cyclic / non-cyclic / Klein group (as appropriate) for one of groups A, B, C for another of groups A, B, C
$A \not\cong B$ $B \not\cong C$ $A \cong C$	B1 (dep*) B1 (dep*) B1 (dep*) **5**	For stating non-isomorphic For stating non-isomorphic For stating isomorphic (with sufficient detail relating to the first 2 marks)
(iii) $\frac{1+2m}{1+2n} \times \frac{1+2p}{1+2q} = \frac{1+2m+2p+4mp}{1+2n+2q+4nq}$ $= \frac{1+2(m+p+2mp)}{1+2(n+q+2nq)} \equiv \frac{1+2r}{1+2s}$	M1* M1 (dep*) A1 A1 **4**	For considering product of 2 distinct elements of this form For multiplying out For simplifying to form shown For identifying as correct form, so closed **SR** $\frac{\text{odd}}{\text{odd}} \times \frac{\text{odd}}{\text{odd}} = \frac{\text{odd}}{\text{odd}}$ earns full credit **SR** If clearly attempting to prove commutativity, allow at most M1
(iv) Closure not satisfied Identity and inverse not satisfied	B1 B1 **2** **13**	For stating closure For stating identity and inverse **SR** If associativity is stated as not satisfied, then award at most B1 B0 *OR* B0 B1